Relations and Functions Exercise 1.2

How to Prove a Function for One to One:

1) We have to prove that

if f(x) = f(y), then x = y

How to Prove a Function for onto (Surjectivity):

1) for every y ∈ B, there exists an element x in A where f(x)=y

How to Prove a Function for Bijectivity

To prove a function is bijective, you need to prove that it is injective and also surjective.
"Injective" means no two elements in the domain of the function gets mapped to the same image. The method has been already described above.
"Surjective" means that any element in the range of the function is hit by the function. The method has been already described above

Question 1

Show that the function f: R∗ → R∗ defined by f(x) = 1/x

is one-one and onto, where R∗ is the set of all non-zero real numbers. Is the result true, if the domain R∗ is replaced by N with co-domain being same as R∗?

Solution

It is given that f: R* → R* is defined by f(x) =1/x

<u>For one – one:</u>

Let x, y ∈ R*

such that f(x) = f(y)

1/x =1/y

⇒ x = y

Therefore, f is one – one.

<u>For onto:</u>

For y ∈ R*

, there exists x =1/y ∈ R*

[as $y \neq 0$] such that

$f(x) = 1/ [1/y]$

$= y$

Therefore, f is onto.

Thus, the given function f is one – one and onto.

Now, consider function g: $N \to R^*$ defined by $g(x) = 1/x$

We have,

$g(x) = g(y)$

$1/x = 1/y$

$x = y$

Therefore, g is one – one.

Further, it is clear that g is not onto as for $1.2 \in R^*$, there does not exit any x in N

such that $g(x) = 1/1.2$

Hence, function g is one-one but not onto.

Question 2

Check the injectivity and surjectivity of the following functions:

(i) f: $N \to N$ given by $f(x) = x^2$

(ii) f: $Z \to Z$ given by $f(x) = x^2$

(iii)f: $R \to R$ given by $f(x) = x^2$

(iv)f: $N \to N$ given by $f(x) = x^3$

(v) f: $Z \to Z$ given by $f(x) = x^3$

Solution

(i) f: $N \to N$ is given by $f(x) = x^2$

It is seen that for $x, y \in N$, $f(x) = f(y) \Rightarrow x^2 = y^2$

$\Rightarrow x = y$.

Therefore, f is injective.

Now, $2 \in N$. But, there does not exist any x in N such that $f(x) = x^2 = 2$.

Therefore, f is not surjective.

Hence, function f is injective but not surjective.

(ii) $f: Z \rightarrow Z$ is given by $f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

Therefore, f is not injective.

Now, $-2 \in Z$. But, there does not exist any element $x \in Z$ such that

$f(x) = -2$ or $x^2 = -2$.

Therefore, f is not surjective.

Hence, function f is neither injective nor surjective.

(iii) $f: R \rightarrow R$ is given by $f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

Therefore, f is not injective.

Now, $-2 \in R$. But, there does not exist any element $x \in R$ such that

$f(x) = -2$ or x

2 = −2.

Therefore, f is not surjective.

Hence, function f is neither injective nor surjective.

(iv) $f: N \rightarrow N$ given by $f(x) = x^3$

It is seen that for $x, y \in N$, $f(x) = f(y) \Rightarrow$

$x^3 = y^3 \Rightarrow x = y$.

Therefore, f is injective.

Now, $2 \in N$. But, there does not exist any element $x \in N$ such that

$f(x) = 2$ or $x^3 = 2$.

Therefore, f is not surjective

Hence, function f is injective but not surjective.

(v) f: Z → Z is given by $f(x) = x^3$

It is seen that for $x, y \in Z$, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

Therefore, f is injective.

Now, $2 \in Z$. But, there does not exist any element $x \in Z$ such that

f(x) = 2 or x

3 = 2.

Therefore, f is not surjective.

Hence, function f is injective but not surjective.

Question 3

Prove that the Greatest Integer Function f: R → R given by f(x) = [x], is neither one – one nor onto, where [x] denotes the greatest integer less than or equal to x.

Solution

f: R → R is given by, f(x) = [x]

It is seen that f(1.2) = [1.2] = 1, f(1.9) = [1.9] = 1.

Therefore, f(1.2) = f(1.9), but 1.2 ≠ 1.9.

Therefore, f is not one – one.

Now, consider $0.7 \in R$.

It is known that f(x) = [x] is always an integer. Thus, there does not exist any

element $x \in R$ such that f(x) = 0.7.

Therefore, f is not onto.

Hence, the greatest integer function is neither one – one nor onto.

Question 4

Show that the Modulus Function f: R → R given by f(x) = |x|, is neither one – one nor onto, where |x| is x, if x is positive or 0 and |x| is − x, if x is negative.

Solution

f: R → R is given by f(x) = |x|

= {x, if x ≥ 0

-x, if x < 0}

It is clear that f (-1) = |-1| = 1 and f (1) = |1| = 1

Therefore, f (−1) = f (1), but −1 ≠ 1.

Therefore, f is not one – one.

Now, consider −1 ∈ R.

It is known that f(x) = |x| is always non-negative. Thus, there does not exist any

element x in domain R such that f(x) = |x| = −1.

Therefore, f is not onto.

Hence, the modulus function is neither one-one nor onto.

Question 5

Show that the Sig-num Function f: R → R, given by

f(x) = {1, if x > 0

0, if x = 0

−1, if x < 0

is neither one-one nor onto.

Solution

f: R → R is given by

f(x) = {1, if x > 0

0, if x = 0

-1, if x < 0

It is seen that f (1) = f(2) = 1, but 1 ≠ 2.

Therefore, f is not one – one.

Now, as f(x) takes only 3 values (1, 0, or −1) for the element −2 in co-domain

R, there does not exist any x in domain R such that f(x) = −2.

Therefore, f is not onto.

Hence, the Sig-num function is neither one – one nor onto.

Question 6

Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one – one.

Solution

It is given that A = {1, 2, 3}, B = {4, 5, 6, 7}.

f: A → B is defined as f = {(1, 4), (2, 5), (3, 6)}.

Therefore, f (1) = 4, f (2) = 5, f (3) = 6

It is seen that the images of distinct elements of A under f are distinct.

Hence, function f is one – one.

Question 7

In each of the following cases, state whether the function is one – one, onto or bijective.

Justify your answer.

(i) f: R → R defined by f(x) = 3 − 4x

(ii) f: R → R defined by f(x) = 1 + x^2

Solution

(i) f: R → R is defined as $f(x) = 3 - 4x$.

Let x, y ∈ R such that f(x) = f(y)

$3-4x = 3-4xy$

$-4x = -4y$

$x = y$

Therefore, f is one – one.

For any real number (y) in R, there exists (3-y)/4 in R such that

$f[(3-y)/4] = 3-[4 (3-y)/4]$

$= y$

Therefore, f is onto.

Hence, f is bijective.

(ii) f: R → R is defined as $f(x) = 1 + x^2$

Let x, y ∈ R such that f(x) = f(y)

$1+ x^2 = 1 + y^2$

$\Rightarrow x = \pm y$

Therefore, f(x) = f(y) does not imply that x1 = x2

For example f (1) = f(-1) = 2

Therefore, f is not one – one.

Consider an element −2 in co-domain R.

It is seen that $f(x) = 1 + x^2$ is positive for all x ∈ R.

Thus, there does not exist any x in domain R such that f(x) = −2.

Therefore, f is not onto.

Hence, f is neither one – one nor onto.

Question 8

Let A and B be sets. Show that f: A × B → B × A such that (a, b) = (b, a) is bijective function.

Solution

f: A × B → B × A is defined as f(a, b) = (b, a).

Let $(a_1, b_1), (a_2, b_2) \in A \times B$ such that

$f(a_1, b_1) = f(a_2, b_2)$

$\Rightarrow (b_1, a_1) = (b_2, a_2)$

$\Rightarrow b_1 = b_2$ and $a_1 = a_2$

$\Rightarrow (a_1, b_1) = (a_2, b_2)$

Therefore, f is one – one.

Now, let (b, a) ∈ B × A be any element.

Then, there exists (a, b) ∈ A × B such that f (a, b) = (b, a). [By definition of f]

Therefore, f is onto.

Hence, f is bijective.

Question 9

Let f: N → N be defined by

f(n) = {(n+1)/2 , if n is odd

n/2, if n is even}

for all n ∈ N

State whether the function f is bijective. Justify your answer.

Solution

It can be observed that:

f (1) = (1+1)/2

= 1

and f (2) =2/2

= 1 [By definition of f(n)]

f (1) = f (2), where 1 ≠ 2

Therefore, f is not one-one.

Consider a natural number (n) in co-domain N.

Case I: n is odd

Therefore, n = 2r + 1 for some r ∈ N. Then, there exists 4r + 1∈ N such that

f (4r + 1) =(4r + 1 + 1)/2

= 2r + 1

Case II: n is even

Therefore, n = 2r for some r ∈ N. Then, there exists 4r ∈ N such that

f(4r) =4r/2

= 2r.

Therefore, f is onto.

Hence, f is not a bijective function.

Question 10

Let A = R − {3} and B = R − {1}. Consider the function f: A → B defined by f(x) = (x−2)/(x−3) Is f one-one and onto? Justify your answer.

Solution

A = R − {3}, B = R − {1} and f: A → B defined by f(x) = (x-2)/(x-3)

Let x, y ∈ A such that f(x) = f(y)

(x-2)/(x-3) =(y-2)/(y-3)

(x – 2)(y – 3) = (y – 2)(x – 3)

xy – 3x – 2y + 6 = xy – 2x – 3y + 6

– 3x – 2y = – 2x – 3y ⇒ x = y

Therefore, f is one-one.

Let $y \in B = R - \{1\}$. Then, $y \neq 1$.

The function f is onto if there exists $x \in A$ such that $f(x) = y$.

Now, $f(x) = y$

$\Rightarrow (x-2)/(x-3) = y$

$\Rightarrow x - 2 = xy - 3y$

$\Rightarrow x(1 - y) = -3y + 2$

$\Rightarrow x = (2-3y)/(1-y) \in A \ [y \neq 1]$

Thus, for any $y \in B$, there exists $(2-3y)/(1-y) \in A$ such that

$f[(2-3y)/(1-y)]$

$= [(2-3y)/(1-y) - 2] / [(2-3y)/(1-y) - 3]$

$= (2-3y-2 + 2y)/(2-3y-3 + 3y) = y$

Therefore, f is onto.

Hence, function f is one – one and onto.

Question 11

Let $f: R \to R$ be defined as $f(x) = x^4$ Choose the correct answer.

(A) f is one-one onto (B) f is many-one onto

(C) f is one-one but not onto (D) f is neither one-one nor onto

Solution

$f: R \to R$ is defined as $f(x) = x^4$

Let $x, y \in R$ such that $f(x) = f(y)$.

$x^4 = y^4$

$x = \pm y$

Therefore, $f(x) = f(y)$ does not imply that $x = y$.

For example f (1) = f(–1) = 1

Therefore, f is not one-one.

Consider an element 2 in co-domain R. It is clear that there does not exist any x in domain R such that f(x) = 2.

Therefore, f is not onto.

Hence, function f is neither one – one nor onto.

The correct answer is D.

Question 12

Let f: R → R be defined as f(x) = 3x. Choose the correct answer.

(A) f is one – one onto (B) f is many – one onto

(C) f is one – one but not onto (D) f is neither one – one nor onto

Solution

f: R → R is defined as f(x) = 3x.

Let x, y ∈ R such that f(x) = f(y).

3x = 3y

x = y

Therefore, f is one-one.

Also, for any real number (y) in co-domain R, there exists y/3 in R such that

f (y/3) =3 (y/3) = y

Therefore, f is onto.

Hence, function f is one – one and onto.

The correct answer is A.