Vector differentiation

Here in this post we will revise our concept of Vector Calculus (differentiation of vectors). This mathematical tool would help us in expressing certain basic ideas with a great convenience while studying electrodynamics.

DIFFERENTIATION OF VECTORS

Consider a vector function f(u) such that

$f(u) = f_x(u)i + f_y(u)j + f_z(u)k$

where f_x, $f_y(u)$ and $f_z(u)$ are scalar functions of u and are components of vector f(u) along x, y, and z directions. If we want to find the derivative of f(u) with respect to u we will have to proceed in the similar manner we used to do with ordinary derivatives thus

$$\frac{df(u)}{du} = \lim_{\Delta u \to 0} \frac{f(u+\Delta u) - f(u)}{\Delta u}$$

where df(u)/du is also a vecor.

Thus in Cartesian coordinated derivative of vector f(u) is given by

$$\frac{d\mathbf{f}}{du} = \frac{df_x}{du}\hat{i} + \frac{df_y}{du}\hat{j} + \frac{df_z}{du}\hat{k}$$

SCALAR AND VECTOR FIELDS

When we talk about fields then in this case a particular scalar or vector quantity is defined not just at a point in the space but it is defined continuously throughout some region in space or maybe the entire region in the space. Now a scalar field ϕ(x,y,z) associates a scalar with each point in the region of space under consideration and a vector field f(x,y,z) associates a vector with each point.

In electrodynamics we will come across the cases where variation in scalar and vector fields from one point to is continuous and is also differentiable in the particular region of space under consideration.

GRADIENT OF A SCALAR FIELD

Consider a scalar field ϕ(x,y,z). This function depends on three variables. Now how would we find the derivative of such functions? If we infinitesimal change dx, dy and dz along x, y, and z axis simultaneously then total differential dϕ of function ϕ(x,y,z) is given as

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

above expression comes from our previous knowledge of partial differentiation.

If we closely examine above equation this could be a result of dot product of two vectors thus,

$$d\varphi = \left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\right) \bullet \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

or,

$d\phi = (\nabla\phi)\bullet(d\mathbf{r})$ where

$$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\right)$$

is gradient of $\phi(x,y,z)$ and gradient of a scalar function is a vector quantity as it is the multiplication of a vector by a scalar.

Thus we see that gradient of any scalar field has both magnitude and direction. Again consider the function $\phi(x,y,z)$ then from ordinary calculas any change in this function as discussed above is given by

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

thus

$d\phi = (\nabla\phi)\bullet(d\mathbf{r}) = |\nabla\phi|\ |d\mathbf{r}|\cos\theta$

From this we see that $d\phi(x,y,z)$ will be maximum when $\cos\theta=1$ which would be the case when $d\mathbf{r}$ would be parallel to $\nabla\phi$. Thus function ϕ changes maximally when one moves in the direction same as that of gradient. So we can say that the direction of $\nabla\phi$ is along the greatest increase of ϕ and the magnitudes of $|\nabla\phi|$ gives the slope along that direction.

CONCLUSION: The gradient $\nabla\phi$ points in the direction of the maximum increase of function $\phi(x,y,z)$ and the magnitude $|\nabla\phi|$ gives the slope or rate of increase along the maximal direction.

THE OPERATOR ∇

While discussing gradient of a scalar function we find that gradient of any function is given by

$$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\right)$$

or,

$$\nabla\varphi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\varphi$$

where the term in parentheses is called "del"

$$\nabla = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right)$$

Del is an vector derivative or vector operator and this operator acts on everything to its right in an expression, until the end of the expression or a closing bracket is reached.

There are three ways in which ∇ can act or operate on a scalar or vector function
1. On a scalar function ϕ : ∇ϕ (the gradient);
2. On a vector functions f, via the dot product: ∇ • v (the divergence);
3. On a vector functions f, via the cross product: ∇ x v (the curl).
Out of these three ways of operation of ∇ on any function we have already discussed gradient of a scalar function.

In this post we learned about scalar and vector fields, gradient of scalar fields and ∇ operator. In the next post we'll learn more about vector differential calculus i.e, in particular we'll discuss divergence and curl of vector fields.
Yes, your comments and queries are heartily invited please do give us a response and comment on our work so that we could further improve it and your response is more than enough to encourage us.