Complex Analysis part 2

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Liouville’s Theorem

If a function $f(z)$ is analytic for all finite values of $z$, and is bounded then it is a constant. Note:- $e^{z+2\pi i} = e^z$

Taylor’s Theorem

If a function $f(z)$ is analytic at all points inside a circle $C$, with its centre at point $a$ and radius $R$ then at each point $z$ inside $C$

$$f(z) = f(a) + (z - a)f'(a) + \frac{1}{2!}(z - a)^2 f''(a) + \ldots + \frac{1}{n!}(z - a)^n f^n(a)$$

Taylor’s theorem is applicable when function is analytic at all points inside a circle.

Laurent Series

If $f(z)$ is analytic on $C_1$ and $C_2$ and in the annular region $R$ bounded by the two concentric circles $C_1$ and $C_2$ of radii $r_1$ and $r_2$ ($r_1 > r_2$) with their centre at $a$ then for all $z$ inside $R$

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots$$

where,

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)dw}{(w-a)^{n+1}}$$

$$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)dw}{(w-a)^{n+1}}$$
Singular points

If a function \( f(z) \) is not analytic at point \( z=a \) then \( z=a \) is known as a singular point or there is a singularity of \( f(z) \) at \( z=a \) for example \( f(z) = \frac{1}{z^2} \) \( z=2 \) is a singularity of \( f(z) \)

Pole of order \( m \)

If \( f(z) \) has singularity at \( z=a \) then from laurent series expansion
\[
f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_m}{(z-a)^m} + \frac{b_{m+1}}{(z-a)^{m+1}}
\]
if \( b_{m+1} = b_{m+2} = 0 \) then
\[
f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_m}{(z-a)^m}
\]
and we say that function \( f(z) \) is having a pole of order \( m \) at \( z=a \). If \( m=1 \) then point \( z=a \) is a simple pole.

Residue

The constant \( b_1 \), the coefficient of \( (z-z_0)^{-1} \), in the Laurent series expansion is called the residue of \( f(z) \) at singularity \( z = z_0 \)
\[
b_1 = Res_{z=z_0}f(z) = \frac{1}{2\pi i} \int_{C_1} f(z)dz
\]

Methods of finding residues

- Residue at a simple pole
  
  if \( f(z) \) has a simple pole at \( z=a \) then \( Resf(a) = \lim_{z\to a} (z-a)f(z) \)

- If \( f(z) = \frac{\Phi(z)}{\Psi(z)} \) and \( \Psi(a) = 0 \) then \( Resf(a) = \frac{\Phi(z)}{\Psi'(z)} \)
Residue at pole of order m

If $f(z)$ is a pole of order m at $z=a$ then

$$Res f(a) = \frac{1}{(m-1)!} \left\{ \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z) \right\}_{z=a}$$

**Residue Theorem**

If $f(z)$ is analytic in closed contour C except at finite number of points (poles) within C, then

$$\int_C f(z)dz = 2\pi i \text{[sum of the residues at poles within C]}$$

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\[1\]This document is created by [http://physicscatalyst.com](http://physicscatalyst.com)