Application of Derivatives Exercise 6.2

Question 1

Show that the function given by f(x) = 3x + 17 is strictly increasing on R.

Solution

f(x) = 3x + 17

Differentiating w.r.t x

f'(x) = 3 > 0, in every interval of R.

Thus, the function is strictly increasing on R.

Question 2

Show that the function given by $f(x) = e^{2x}$ is strictly increasing on R.

Solution

$f(x) = e^{2x}$

Differentiating w.r.t x

$f'(x) = 2e^{2x} > 0$, in every interval of R.

Hence, f is strictly increasing on R.

Question 3

Show that the function given by f(x) = sin x is

(a) strictly increasing in (0 ,π/2)

(b) strictly decreasing in (π/2, π)

(c) neither increasing nor decreasing in (0, π)

Solution

The given function is f(x) = sin x.

Differentiating w.r.t x

f'(x) = cosx

(a) Since for each (0 ,π/2) we have cos x > 0

Hence, f is strictly increasing in (0 ,π/2)

(b) Since for each (π/2, π) , we have cos x < 0

Hence, f is strictly decreasing in (π/2, π)

(c) From the results obtained in (a) and (b), it is clear that f is neither increasing nor decreasing in (0, π).

Question 4

Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is

(a) strictly increasing (b) strictly decreasing

Solution

$f(x) = 2x^2 - 3x$

Differentiating w.r.t x

f'(x) =4x-3

Equating f'(x) =0, we get

4x-3 =0

x= ¾

The point *x* = 3/4 divides the real line into two disjoint intervals, namely, (– ∞, 3/4), (3/4, ∞)
In interval (– ∞, 3/4), f'(x) < 0

Hence, the given function (f) is strictly decreasing in interval (– ∞, 3/4)

In interval (3/4, ∞) , f'(x) > 0
Hence, the given function (f) is strictly increasing in interval (3/4, ∞)

Question 5

Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

(a) strictly increasing (b) strictly decreasing

Solution

$f(x) = 2x^3 - 3x^2 - 36x + 7$

Differentiating w.r.t x

$f'(x) = 6x^2 - 6x - 36$

Equating $f'(x) = 0$, we get

$6x^2 - 6x - 36 = 0$

Or $x = -2, 3$

The points $x = -2$ and $x = 3$ divide the real line into three disjoint intervals i.e., $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$

In intervals is positive $(-\infty, -2)$, $(3, \infty)$, $f'(x) > 0$

while in interval $(-2, 3)$, is negative, $f'(x) < 0$

Hence, the given function (f) is strictly increasing in intervals $(-\infty, -2)$, $(3, \infty)$

, while function (f) is strictly decreasing in interval $(-2, 3)$,

Question 6

Find the intervals in which the following functions are strictly increasing or decreasing:

(a) $x^2 + 2x - 5$

(b) $10 - 6x - 2x^2$

(c) $-2x^3 - 9x^2 - 12x + 1$

(d) $6 - 9x - x^2$

(e) $(x + 1)^3(x - 3)^3$

Solution

(a) We have,

$F(x) = x^2 + 2x - 5$

Differentiating w.r.t x

$f'(x) = 2x+2$

Equating $f'(x) = 0$, we get

$2x+2=0$

$x = -1$

Point $x = -1$ divides the real line into two disjoint intervals i.e., $(-\infty, -1)$ and $(-1, \infty)$

In interval $(-\infty, -1)$, $f'(x) < 0$

f is strictly decreasing in interval $(-\infty, -1)$

In interval $(-1, \infty)$, $f'(x) > 0$

f is strictly increasing in interval $(-1, \infty)$

(b) We have,

$f(x) = 10 - 6x - 2x^2$

Differentiating w.r.t x

$f'(x) = -6-4x$

Equating $f'(x) = 0$, we get

$-6-4x=0$

$x=-3/2$

The point divides the real line into two disjoint intervals i.e., $(-\infty, -3/2)$ and $(-3/2, \infty)$

In interval $(-\infty, -3/2)$, $f'(x) > 0$

f is strictly increasing for $(-\infty, -3/2)$

In interval $(-3/2, \infty)$, $f'(x) < 0$

f is strictly decreasing for $(-3/2, \infty)$

(c) We have,

$f(x) = -2x^3 - 9x^2 - 12x + 1$

Differentiating w.r.t x

$f'(x) = -6x^2 - 18x - 12 = -6(x+1)(x+2)$

Equating $f'(x) = 0$, we get

$-6x^2 - 18x - 12 = 0$

$x^2 + 3x + 2 = 0$

$(x+1)(x+2) = 0$

Or $x = -1, -2$

Points $x = -1$ and $x = -2$ divide the real line into three disjoint intervals i.e., $(-\infty, -2)$, $(-2, -1)$ and $(-1, \infty)$

In intervals, i.e., when $x < -2$ and $x > -1$, $f'(x) < 0$

f is strictly decreasing for $x < -2$ and $x > -1$.

Now, in interval $(-2, -1)$ i.e., when $-2 < x < -1$, $f'(x) > 0$

f is strictly increasing for $-2 < x < -1$

(d) We have,

$F(x) = 6 - 9x - x^2$

Differentiating w.r.t x

$f'(x) = -9 - 2x$

Equating $f'(x) = 0$, we get

$x = -9/2$

The point divides the real line into two disjoint intervals i.e., $(-\infty, -9/2)$ and $(-9/2, \infty)$

In interval $(-\infty, -9/2)$, $f'(x) > 0$

f is strictly increasing for $(-\infty, -9/2)$

In interval $(-9/2, \infty)$, $f'(x) < 0$

f is strictly decreasing for $(-9/2, \infty)$

(e) We have,

$f(x) = (x + 1)^3(x - 3)^3$

Differentiating w.r.t x

$f'(x) = 3(x+1)^2(x-3)^3 + 3(x+1)^3(x-3)^2$

$= 3(x+1)^2(x-3)^2 [x-3 + x+1]$

$= 6(x+1)^2(x-3)^2 (x-1)$

Equating f'(x) =0, we get

x = -1 or 3 or 1

The points x = −1, x = 1, and x = 3 divide the real line into four disjoint intervals

i.e. $(-\infty, -1)$, $(-1, 1)$, $(1, 3)$, and $(3, \infty)$

In intervals and $(-1, 1)$, $(-\infty, -1)$, $f'(x) < 0$

f is strictly decreasing in intervals and (−1, 1).

In intervals $(1, 3)$ and $(3, \infty)$, $f'(x) > 0$

f is strictly increasing in intervals $(1, 3)$ and $(3, \infty)$

Question 7

Show that y = log(1+x) -2x/(2+x) x > 1 is an increasing function of x throughout its domain.

Solution

We have,

y = log(1+x) -2x/(2+x)

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{1}{1+x} - \frac{(2+x)2-2x(1)}{(2+x)^2} = \frac{x^2}{(2+x)^2}$$

Equating dy/dx =0, we get

$x^2/(2+x)^2 = 0$

or x =0 as x >-1

Since x > −1, point x = 0 divides the domain (−1, ∞) in two disjoint intervals i.e., −1 < x < 0 and x > 0.

When −1 < x < 0, we have dy/dx > 0

Also, when x > 0, we have dy/dx > 0

Hence, function f is increasing throughout this domain.

Question 8

Find the values of x for which $y = [x(x-2)]^2$ is an increasing function.

Solution

We have,

$y = [x(x-2)]^2$

$y = (x^2 - 2x)^2 = x^4 + 4x^2 - 4x^3$

Differentiating w.r.t x

$dy/dx = 4x^3 + 8x - 12x^2$

Equating dy/dx =0, we get

$4x(x^2 + 2 - 3x) = 0$

$4x(x-1)(x-2) = 0$

The points x = 0, x = 1, and x = 2 divide the real line into four disjoint intervals i.e.,

$(-\infty,0),(0,1),(1,2)\ (2,\infty)$

In intervals $(-\infty,0),(1,2)$, dy/dx < 0

y is strictly decreasing in intervals, dy/dx

However, in intervals (0, 1) and (2, ∞), dy/dx > 0

y is strictly increasing in intervals (0, 1) and (2, ∞).

Question 9

Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in [0, π/2]

Solution

We have,

$$y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$$

Differentiating w.r.t θ

$$\frac{dy}{d\theta} = \frac{(2+\cos\theta)\,(4\cos\theta)-(4\sin\ \theta)(-\sin\theta)}{(2+\cos\theta)^2} - 1 = \frac{8\cos\theta+4}{(2+\cos\theta)^2} - 1$$

Equating dy/d θ =0, we get

$$\frac{8\cos\theta+4}{(2+\cos\theta)^2} - 1 = 0$$

$8\cos\theta + 4 = (2+\cos\theta)^2$

$8\cos\theta + 4 = 4 + 4\cos\theta + \cos^2\theta$

$\cos^2\theta - 4\cos\theta = 0$

$\cos\theta(\cos\theta - 4)$

Since cos θ ≠ 4, cos θ = 0.

Or θ= π/2

Now,

$$\frac{dy}{d\theta} = \frac{8\cos\theta+4}{(2+\cos\theta)^2} - 1 = \frac{8\cos\theta+4-(4+cos^2\theta+4cos\theta)}{(2+\cos\theta)^2} = \frac{4\cos\theta-cos^2\theta}{(2+\cos\theta)^2} = \frac{\cos\theta(4-cos\theta)}{(2+cos\theta)^2}$$

In interval (0, π/2) we have cos θ > 0. Also, 4 > cos θ, 4 – cos θ > 0

So dy/dθ > 0

Therefore, y is strictly increasing in interval (0, π/2)

Also, the given function is continuous at 0 and π/2

Hence, y is increasing in interval [0, π/2]

Question 10

Prove that the logarithmic function is strictly increasing on (0, ∞).

Solution

F(x) = log x

Differentiating w.r.t x

f'(x) =1/x

It is clear that for x > 0, f'(x) > 0

Hence, f(x) = log x is strictly increasing in interval (0, ∞).

Question 11

Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor strictly decreasing on (−1, 1).

Solution

$f(x) = x^2 - x + 1$

Differentiating w.r.t x

f'(x) =2x -1

Equating f'(x) =0, we get

2x-1 = 0

x= 1/2

The point divides the interval (−1, 1) into two disjoint intervals (-1, ½) and (½,1)

Now, in interval (-1, ½), f'(x) =2x -1 < 0

Therefore, f is strictly decreasing in interval (-1, ½)

However, in interval (½,1), f'(x) =2x -1 > 0

Therefore, f is strictly increasing in interval (½,1)

Hence, f is neither strictly increasing nor decreasing in interval (−1, 1).

Question 12

Which of the following functions are strictly decreasing on (0, π/2)?

(A) cos x (B) cos 2x (C) cos 3x (D) tan x

Solution

(A) Let f(x) =cos x

f'(x) = -sin x

In interval (0, π/2), f'(x) < 0

So, f(x) is strictly decreasing in interval (0, π/2)

(B) Let

F(x) = cos 2x

f'(x) = -2 sin 2x

for x in (0, π/2), 2x would in (0, π), So sin2x > 0

and f'(x) = -2 sin 2x < 0

So, cos 2x is strictly decreasing in interval (0, π/2)

(C) Let f(x) = cos 3x

f'(x) = -3 sin 3x

Equating f'(x)=0

Sin 3x =0

Or x = π/3 as x in (0, π/2)

The point divides the interval into two disjoint intervals (0, π/3) and (π/3, π/2)

For interval (0, π/3), f'(x) = -3 sin 3x < 0

So, it is strictly decreasing in interval (0, π/3)

For interval (π/2, π/3), f'(x) = -3 sin 3x > 0

So, it is strictly increasing in interval (π/2, π/3), Hence, function is neither increasing nor decreasing in interval (0, π/2)

(D) Let

f(x) = tan x

$f'(x) = \sec^2 x$

for interval (0, π/2), f'(x) > 0

So, function is strictly increasing in interval (0, π/2)

Therefore, functions cos x and cos 2x are strictly decreasing in (0, π/2)

Hence, the correct answers are A and B.

Question 13

On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ strictly

decreasing?

(A) (0,1)

(B) (π/2, π)

(C) (0, π/2)

(D) None of these

Solution

We have,

$f(x) = x^{100} + \sin x - 1$

$f'(x) = 100x^{99} + \cos x$

In interval (0,1), f'(x) > 0 as cos x > 0 and $100x^{99} > 0$

Thus, function f is strictly increasing in interval (0, 1).

In interval (π/2, π), cos x < 0 and $100x^{99} > 0$ and $100x^{99} > \cos x$, So f'(x) > 0

Thus, function f is strictly increasing in interval (π/2, π)

In interval (0, π/2), cos x > 0 and $100x^{99} > 0$, So f'(x) > 0

f is strictly increasing in interval (0, π/2)

Hence, function f is strictly decreasing in none of the intervals.

The correct answer is D.

Question 14

Find the least value of a such that the function f given $f(x) = x^2 + ax + 1$ is strictly increasing on (1, 2).

Solution

We have,

$f(x) = x^2 + ax + 1$

$f'(x) = 2x + a$

Now, function f will be increasing in (1, 2), if in (1, 2), $f'(x) > 0$

$2x + a > 0$

$2x > -a$

$x > -a/2$

Now x lies in (1,2), So the least value of a such that

$-a/2 = 1$

$a = -2$

Hence, the required value of a is −2.

Question 15

Let I be any interval disjoint from (−1, 1). Prove that the function f given by

$F(x) = x + 1/x$ is strictly increasing on I.

Solution

$f(x) = x + 1/x$

$f'(x) = 1 - 1/x^2$

$1 - 1/x^2 = 0$

$x^2 = 1$

or $x = +1, -1$

The points x = 1 and x = −1 divide the real line in three disjoint intervals i.e., $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$

In interval (−1, 1), it is observed that

$x^2 < 1$

$1/x^2 > 1$

Or $0 > 1 - 1/x^2$ $(x \neq 0)$

f is strictly decreasing on (−1, 1)

In intervals $(-\infty, -1)$ and $(1, \infty)$, it is observed that:

$x^2 > 1$

$1/x^2 < 1$

$1 - 1/x^2 > 0$

f is strictly increasing on $(-\infty, -1)$ and $(1, \infty)$

Hence, function f is strictly increasing in interval I disjoint from (−1, 1).

Hence, the given result is proved.

Question 16

Prove that the function f given by f(x) = log sin x is strictly increasing on (0, π/2)

And strictly decreasing on (π/2, π)

Solution

f(x) = log sin x

f′(x) = (1/sin x) cos x = cot x

In interval (0, π/2), f′(x) =cot x > 0

f is strictly increasing in (0, π/2)

In interval (π/2, π), f′(x) =cot x < 0

f is strictly decreasing in (π/2, π)

Question 17

Prove that the function f given by f(x) = log cos x is strictly decreasing on (0, π/2)

And strictly increasing on (π/2, π)

Solution

f(x) = log cos x

f'(x) = (1/cos x) (-sin x) = -tan x

In interval (0, π/2), f'(x) = -tan x < 0

f is strictly decreasing on (0, π/2)

In interval (π/2, π), f'(x) = -tan x > 0

f is strictly increasing on (π/2, π)

Question 18

Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in R.

Solution

$f(x) = x^3 - 3x^2 + 3x - 100$

$f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$

For any x in R, $(x - 1)^2 > 0$.

Hence, the given function (f) is increasing in R.

Question 19

The interval in which $y = x^2 e^{-x}$ is increasing is

(A) (– ∞, ∞)

(B) (−2, 0)

(C) (2, ∞)

(D) (0, 2)

Solution

$y = x^2 e^{-x}$

$dy/dx = 2x e^{-x} – x^2 e^{-x} = xe^{-x}(2-x)$

for dy/dx=0

x= 0 or x =2

The points x = 0 and x = 2 divide the real line into three disjoint intervals

i.e., $(-\infty,0)$, $(0 ,2)$ and $(2, \infty)$

In intervals $(-\infty,0)$,$(2, \infty)$, $dy/dx < 0$

f is decreasing on $(-\infty,0)$,$(2, \infty)$,

In interval (0, 2), $dy/dx > 0$

f is strictly increasing on (0, 2).

Hence, f is strictly increasing in interval (0, 2).

The correct answer is D.